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Accelerated coloured particles in general relativity

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Abstract. Trautman's accelerated coloured particle solution of the Yang–Mills field equations are shown also to exist in general relativity.

It is well known that accelerated charged particles in classical electromagnetism loose their energy and momentum by radiation. Recently, Trautman (1981) pointed out that the situation might be quite different in the classical Yang–Mills theory. He conjectures that a single classical, colour-carrying particle radiates if the colour remains constant. Acceleration, and hence the radiation, vanishes if the colour changes with time.

In this paper we report that Trautman's observations are valid in general relativity as well. Following Newman and Posadas (1969) the potential one-form, the field two-form and the line element corresponding to an accelerated charged particle, respectively, are

$$\mathfrak{A} = (e/r)[\mathrm{d}r + (\varepsilon - Qr)\,\mathrm{d}u] \tag{1}$$

$$f = \mathrm{d}\mathfrak{A} = (\varepsilon/r^2)\,\mathrm{d}u \wedge \mathrm{d}r + e\,\mathrm{d}u \wedge \mathrm{d}Q \tag{2}$$

$$\mathrm{d}s^2 = 2(K - Qr + \psi_2^0/r + E^2/r^2)\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r - (r^2/2P^2)\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} \tag{3}$$

where e is charge, ε is the norm of the four-velocity of the particle and $Q = P^{-1}\dot{P} = P^{-1}\,\partial P/\partial u$; K, ψ_2^0, E and Q satisfy the equations

$$\begin{aligned} &\dot{\psi}_2^0 - 3\psi_2^0 Q = \eth\eth^* K + 4P^2 Q_{,\zeta} Q_{,\bar{\zeta}} \\ &\dot{E} - 2EQ = -2\eth(PQ_{,\zeta}) \qquad \eth\eth^*\psi_2^0 = -2E(\dot{E} - 2EQ) \end{aligned} \tag{4}$$

with $\eth\eth^* = 4P^2\partial_\zeta\partial_{\bar{\zeta}}$. These solutions are known as the Robinson–Trautman (1962) metrics in general relativity. Now let the gauge potential one-form A be given as

$$A = q(u)\mathfrak{A} \tag{5}$$

where $q(u)$ is a matrix valued function of u and $\mathfrak{A}$ is given in (1) and the gauge field two-form written by

$$F = r^{-1}(\dot{q} + \varepsilon q/r)\,\mathrm{d}u \wedge \mathrm{d}r + q\,\mathrm{d}u \wedge \mathrm{d}Q. \tag{6}$$

The Yang–Mills field equations

$$\mathrm{d}^*F + A \wedge {}^*F - {}^*F \wedge A = {}^*J \tag{7}$$

are satisfied exactly if and only if (i) $\dot{q} = 0$. Colour remains constant but since $Q \neq 0$

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then the particles radiate, or (ii) $\dot{q} \neq 0$. $q = \alpha u + \beta$, where α and β are constant matrices satisfying

$$\alpha + [\alpha, \beta] = 0 \tag{8}$$

and $Q = 0$ which means there is no radiation. Hence the metric (3) and the potential (5) with the constraints (i), or (ii) with (8) solve the Einstein–Yang–Mills field equations. It can be shown easily that the matrices α and β satisfy

$$\alpha^n = 0 \qquad \text{for} \quad n \geq 2 \tag{9}$$

$$\operatorname{Tr}(\alpha^n \beta^m) = 0 \qquad \text{for} \quad n \geq 1 \qquad m \geq 0 \tag{10}$$

hence

$$\operatorname{Tr} q^n = \operatorname{Tr} \beta^n = \text{constant}. \tag{11}$$

These last properties imply that the energy–momentum tensors corresponding to Maxwell and Yang–Mills fields become identical (assuming $\operatorname{Tr} q^2 = 1$). If (for case (ii)) $\varepsilon = 0$, corresponding to massless coloured particles, since $Q = 0$ as well, then the energy–momentum tensor vanishes but the field is different from zero

$$F = (\alpha / r)\, \mathrm{d}u \wedge \mathrm{d}r \tag{12}$$

and the metric becomes flat

$$\mathrm{d}s^2 = 2\, \mathrm{d}u\, \mathrm{d}r - (r^2/2P^2)\, \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta}.$$

The same potential (5) with $Q = 0$ and the Bertotti–Robinson metric also solve the Einstein–Yang–Mills field equations with the same conclusions (i) or (ii).

The form of the gauge potential in (5) suggests the following generalisation. Let $\sigma = (l, n, m, \bar{m})$ and $\mathfrak{A}$ be the tetrad and the electromagnetic potential one-forms, respectively, representing a solution of the Einstein–Maxwell field equations; then σ and $q(u)\mathfrak{A}$ are solutions of the Einstein–Yang–Mills field equations, where $q(u)$ is a matrix valued function of the null coordinate u, if and only if

$$\text{(i)} \quad \dot{q} = 0 \qquad \text{or} \qquad \text{(ii)} \quad \dot{q} \neq 0 \qquad q = \alpha u + \beta \tag{13}$$

where α and β are constant matrices satisfying

$$\alpha + [\alpha, \beta] = 0 \tag{14}$$

and, using the Newman–Penrose (1962) formalism

$$\begin{aligned} &D\mathfrak{A}_l - (\rho + \bar{\rho})\mathfrak{A}_l + \kappa \mathfrak{A}_{\bar{m}} + \bar{\kappa} \mathfrak{A}_m - \mathfrak{A}_l^2 = 0 \\ &D\mathfrak{A}_{\bar{m}} - (\pi + \bar{\tau})\mathfrak{A}_l - (\rho - 2\varepsilon)\mathfrak{A}_{\bar{m}} + \bar{\sigma}\mathfrak{A}_m + \bar{\phi}_0 + \mathfrak{A}_l \mathfrak{A}_{\bar{m}} = 0 \\ &\Delta\mathfrak{A}_l + \bar{\delta}\mathfrak{A}_m + (\mu + \bar{\mu})\mathfrak{A}_l + \delta \mathfrak{A}_{\bar{m}} + (\tau - 2\beta)\mathfrak{A}_{\bar{m}} \\ &\quad + (\bar{\tau} - 2\bar{\beta})\mathfrak{A}_m - (\phi_1 + \bar{\phi}_1) - (\mathfrak{A}_l \mathfrak{A}_n - 2\mathfrak{A}_m \mathfrak{A}_{\bar{m}}) = 0 \end{aligned} \tag{15}$$

with $l = \mathrm{d}u$ and

$$\mathfrak{A} = \mathfrak{A}_n l + \mathfrak{A}_l n - \mathfrak{A}_m \bar{m} - \mathfrak{A}_{\bar{m}} m \tag{16}$$

and ϕ_l are the electromagnetic field spinors. In this general case the energy–momentum tensors of the Maxwell and the Yang–Mills fields are also identical, hence two different fields which are not gauge equivalent define the same geometry. This is an example indicating that there can be no Birkhoff theorem in the Einstein–Yang–Mills system.

We may fix the direction of q by a gauge transformation of the form

$$\omega(u) = I + \alpha u \tag{17}$$

but the potential one-form is translated by an amount $\alpha \, \mathrm{d}u$

$$A = \beta \mathfrak{A} + \alpha \, \mathrm{d}u \tag{18}$$

and the field two-form becomes

$$F = \alpha \, \mathrm{d}u \wedge \mathfrak{A} + \beta f \tag{19}$$

where $f = \mathrm{d}\mathfrak{A}$. The gauge transformation (17) may change the u dependence of the field F but it changes its asymptotic form as well.

Finally, we should like to remark that two gauge equivalent Maxwell potentials $\mathfrak{A}$ and $\mathfrak{A} + \mathrm{d}\gamma$ may not necessarily provide gauge equivalent Yang–Mills potentials. As an example let $\mathfrak{A} = \mathrm{d}\gamma$ (pure gauge); then the corresponding (this is valid only in case (ii)) Yang–Mills field is

$$F = \alpha \, \mathrm{d}u \wedge \mathrm{d}\gamma \tag{20}$$

which has no energy-momentum tensor, hence the space–time is described by a vacuum solution. These kinds of solutions are known as vacuum solutions (Casalbuoni *et al* 1979).

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References

Casalbuoni R, Domokos G and Kovesi-Domokos S 1979 *Phys. Lett.* **81B** 34
Newman E and Penrose R 1962 *J. Math. Phys.* **3** 566
Newman E T and Posadas R 1969 *Phys. Rev.* **187** 1784
Robinson I and Trautman A 1962 *Proc. R. Soc.* **265** 463
Trautman A 1981 *Phys. Rev. Lett.* **46** 875